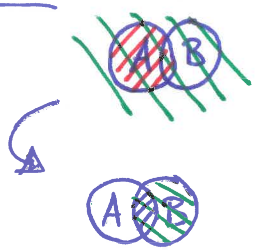
§2.4 Conditional Probability

Conditional probability is about restricting the sample space.

It is a simple but very useful & important application.

Def: $P(A \mid B)$ "the probability of A given B"
is the probability of the event A restricted to the sample space B

Recall: If S is an equiprobable space, then



$$P(A) = \frac{\#(A)}{\#(S)}$$

$$P(A \mid B) = \frac{\#(A \cap B)}{\#(B)}$$

More generally:

Formula: $$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Alternate form: $P(A \cap B) = P(A \mid B) \cdot P(B)$

Conditional probability problems are often easiest to visualize if we express them either in form of table or decision tree (next page)

Three standard applications:

- Restricting a sample space to remove some outcomes (due to new information).

  Example: A system has two (independent) safety features. #1 fails with probability $1/3$
  #2 fails with probability $1/5$

  If a safety fails, what is the probability it was #1?

  $\Rightarrow P(\text{Safety } 1 \mid \text{Safety } 1 \text{ or Safety } 2)$

- Working with a sample space described as a combination of restricted sample spaces.

  Example: System components are 20% from company A and 80% from company B. The failure rate of components from A is 10% & the failure rate of components from B is 1%. What is probability that a component in the system will fail?

  $\Rightarrow$ Know $\begin{cases} P(\text{fail} \mid A) \\ P(\text{fail} \mid B) \end{cases}$ Want $P(\text{fail})$

- Reversing conditional probabilities (combination of previous two applications)

  Example 1: Like before, system is 20% A & 80% B. A fails at 10% & B fails at 1%. If something fails, what is probability it was from A?

  $\Rightarrow$ Know $\begin{cases} P(\text{fail} \mid A) \\ P(\text{fail} \mid B) \end{cases}$ Want $P(A \mid \text{fail})$

Reversing conditional probabilities (continued)

Example 2: A test incorrectly returns a positive result ("false positive") with probability 5% and incorrectly returns a negative result ("false negative") with probability 1%. If probability of being positive is 10% then what is probability of being positive when test says positive?

$\Rightarrow$ Know
$$\begin{cases} P(\text{false pos}) = P\left(\begin{smallmatrix}\text{test}\\ \text{positive}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{actually}\\ \text{negative}\end{smallmatrix}\right) = 5\% \\ P(\text{false neg}) = P\left(\begin{smallmatrix}\text{test}\\ \text{negative}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{actually}\\ \text{positive}\end{smallmatrix}\right) = 1\% \end{cases}$$

Want $P\left(\begin{smallmatrix}\text{actually}\\ \text{positive}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\text{test}\\ \text{positive}\end{smallmatrix}\right)$

Three ways to do computations with conditional prob.

- Formulas $\quad P(A\mid B) = \dfrac{P(A\cap B)}{P(B)}$
- Tables
- Trees

Focus on these because they are quite simple.

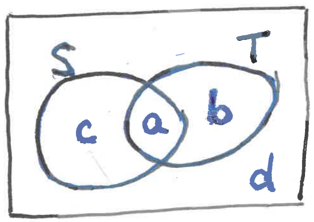
Tables

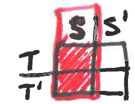
Suppose we have two events S & T

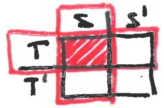
We can organize counts of #outcomes in table

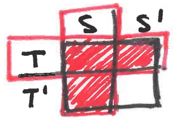
"not S" / "not T"

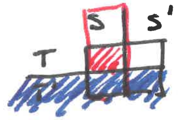
#	S	S'	Total
T	a	b	a+b
T'	c	d	c+d
Total	a+c	b+d	a+b+c+d

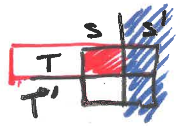


S is  $\#S = a+c \qquad P(S) = \dfrac{a+c}{a+b+c+d}$

$S\cap T$ is  $\#(S\cap T) = a \qquad P(S\cap T) = \dfrac{a}{a+b+c+d}$

$S\cup T$ is  $\#(S\cup T) = a+b+c \qquad P(S\cup T) = \dfrac{a+b+c}{a+b+c+d}$

$S\mid T$ is  Throw this part away $\qquad P(S\mid T) = \dfrac{a}{a+b}$

$T\mid S$ is  $\qquad P(T\mid S) = \dfrac{a}{a+c}$

Example Add 2 dice $\quad S = \{\text{multiple of } 4\}$, $T = \{\text{multiple of } 3\}$

Table:

#	4x	not 4x	Total
3x	1	11	12
not 3x	8	16	24
Total	9	27	36

$P(4x) = \dfrac{9}{36} \qquad P(4x\mid 3x) = \dfrac{1}{12}$

$P(4x\cap 3x) = \dfrac{1}{36} \qquad P(3x\mid 4x) = \dfrac{1}{9}$

$P(4x\cup 3x) = \dfrac{1+11+8}{36} = \dfrac{20}{36} \qquad P(3x\mid \text{not } 4x) = \dfrac{11}{27}$

Note: If we divide each cell of table by 36 then we have table of Probability instead of Count

Now let's apply this to solve an actual problem:

Example 1. System is 20% from company A and 80% from company B. Company A parts fail at 10% rate and B at 1%. If a part fails what is the probability that it was from company A?

Know: $P\left(\begin{smallmatrix}\text{part is}\\ \text{from A}\end{smallmatrix}\right) = 20\%$ $\quad P(\text{fail} \mid \text{from A}) = 10\%$

$P\left(\begin{smallmatrix}\text{part is}\\ \text{from B}\end{smallmatrix}\right) = 80\%$ $\quad P(\text{fail} \mid \text{from B}) = 1\%$

Table:

Prob	from A	from B	Total
fail	① 2%	② .8%	2.8%
not fail	③ 18%	④ 79.2%	97.2%
Total	20%	80%	100%

P(from A) P(from B)

① $P(\text{from A} \cap \text{fail}) = P(\text{from A}) \cdot P(\text{fail} \mid \text{from A})$
$= 20\% \cdot 10\% = 2\%$

② $P(\text{from B} \cap \text{fail}) = P(\text{from B}) \cdot P(\text{fail} \mid \text{from B})$
$= 80\% \cdot 1\% = .8\%$

Don't need to calculate these for this problem

③ $P(\text{from A} \cap \text{not fail}) = P(\text{from A}) - P(\text{from A} \cap \text{fail})$
$= 20\% - 2\% = 18\%$

④ $P(\text{from B} \cap \text{not fail}) = P(\text{from B}) - P(\text{from B} \cap \text{fail})$
$= 80\% - .8\% = 79.2\%$

Totals $2\% + .8\% = 2.8\%$ $\quad 18\% + 79.2\% = 97.2\%$

Answer: $P(\text{from A} \mid \text{fail}) = \dfrac{2\%}{2.8\%} = 71.4\%$

Example 2. False positive rate is 5% & false negative is 1%. If probability of positive is 10% then what is probabity of being positive if test says positive?

Know: $P\left(\begin{smallmatrix}\text{actually}\\ \text{positive}\end{smallmatrix}\right) = 10\%$ $\quad P\left(\begin{smallmatrix}\text{test}\\ \text{positive}\end{smallmatrix} \middle| \begin{smallmatrix}\text{actually}\\ \text{negative}\end{smallmatrix}\right) = 5\%$

$P\left(\begin{smallmatrix}\text{test}\\ \text{negative}\end{smallmatrix} \middle| \begin{smallmatrix}\text{actually}\\ \text{positive}\end{smallmatrix}\right) = 1\%$

Table:

		Actually		
	Prob	Positive	Negative	Total
Test	Positive	④ 9.9%	② 4.5%	14.4%
	Negative	③ .1%	⑤ 85.5%	85.6%
	Total	10%	① 90%	100%

① $P\left(\begin{smallmatrix}\text{actually}\\ \text{negative}\end{smallmatrix}\right) = 100\% - P\left(\begin{smallmatrix}\text{actually}\\ \text{positive}\end{smallmatrix}\right)$
$= 100\% - 10\% = 90\%$

② $P\left(\begin{smallmatrix}\text{actually}\\ \text{neg}\end{smallmatrix} \cap \begin{smallmatrix}\text{test}\\ \text{pos}\end{smallmatrix}\right) = P\left(\begin{smallmatrix}\text{actually}\\ \text{neg}\end{smallmatrix}\right) \cdot P\left(\begin{smallmatrix}\text{test}\\ \text{pos}\end{smallmatrix} \middle| \begin{smallmatrix}\text{actually}\\ \text{neg}\end{smallmatrix}\right)$
$= 90\% \cdot 5\% = 4.5\%$

③ $P\left(\begin{smallmatrix}\text{actually}\\ \text{pos}\end{smallmatrix} \cap \begin{smallmatrix}\text{test}\\ \text{neg}\end{smallmatrix}\right) = P\left(\begin{smallmatrix}\text{actually}\\ \text{pos}\end{smallmatrix}\right) \cdot P\left(\begin{smallmatrix}\text{test}\\ \text{neg}\end{smallmatrix} \middle| \begin{smallmatrix}\text{actually}\\ \text{pos}\end{smallmatrix}\right)$
$= 10\% \cdot 1\% = .1\%$

(Subtract to get ④ & ⑤)

Totals $9.9\% + 4.5\% = 14.4\%$ $\quad .1\% + 85.5\% = 85.6\%$

Answer: $P\left(\begin{smallmatrix}\text{actually}\\ \text{pos}\end{smallmatrix} \middle| \begin{smallmatrix}\text{test}\\ \text{pos}\end{smallmatrix}\right) = \dfrac{9.9\%}{14.4\%} = 68.8\%$